Edge Colouring and the Minimum Number of Colours such that Every Complete Geometric Graph on n Vertices has an Edge Colouring such that Disjoint Edges get Distinct Colours.

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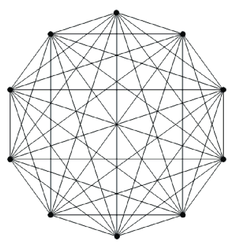
Abstract

We will be edge coloring complete geometric graphs, looking to find the minimum number of colors such that every complete geometric graph on n Vertices has an edge colouring such that crossing edges and disjoint edges get distinct colours.

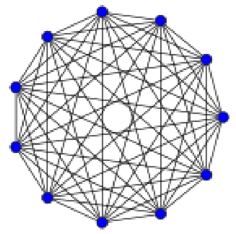
Introduction

A complete geometric graph can be defined by letting P be a set of n points in the plane with no three collinear. Draw a straight line-segment between each pair of points in P. We obtain the complete geometric graph with vertex set P, denoted by K_P.

Here is an example of complete geometric graphs K_{10} and K_{11}



K_{10} Complete Geometric Graph



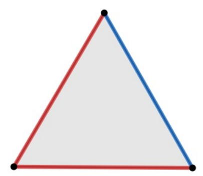
K_{11} Complete Geometric Graph

Disjoint Edges

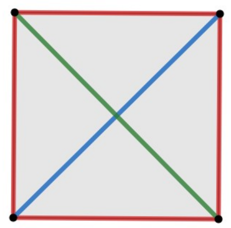
Two edges K_P are disjoint if they do not intersect. Let $D(n)$ be the minimum number of colours for disjoint edges. Here, edges receiving the same colour must intersect. So each colour class is a Geometric Thrackle. Since there are point sets for which $\frac{n}{2}$ edges are Pairwise Disjoint, $D(n) \geq \frac{n}{2}$. So $D(n) \leq n - 1$. The conjecture to go along with this definition is $D(n) \leq (1 - \epsilon)n$ for some $\epsilon < 0$.

Upper bound

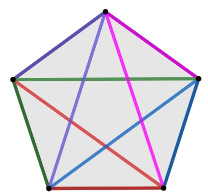
In Nikita Chernega, Alexandr Polyanskii, and Rinat Sadykov's article on disjoint edges in geometric graphs, they show that for a geometric graph with n vertices and e edges there are at least $\frac{n}{2}\binom{2e/n}{3}$ pairs of disjoint edges provided that $2e \geq n$ and all the vertices of the graph are pointed. This result is what I based my research off of because with in this same article they also prove that if any edge of a geometric graph with n vertices is disjoint from at most m edges, then the number of edges of this graph does not exceed $n(\sqrt{1 + 8m} + 3)/4$ provided that n is sufficiently large. Looking at these two results and the upper bound that was found by Bose et al, $A(n) \leq n - \sqrt{\frac{n}{12}}$ for pairwise crossing edges.



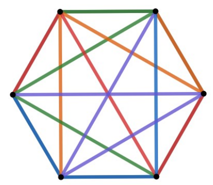
K_3



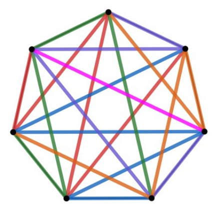
K_4



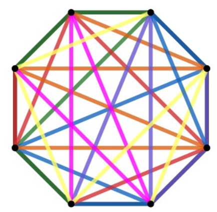
K_3



K_4



K_3



K_4

Results

Drawing all six graphs by hand most likely took away odds of making graphs with a better minimum. For all graphs drawn, ,with the exception of K_5 and K_6 they stayed true to the definition, that being that since there are point sets for which $\frac{n}{2}$ edges are Pairwise Disjoint, $D(n) \geq \frac{n}{2}$. So $D(n) \leq n - 1$. For K_5 and K_6, they both use the same number of colors as the number of vertices that they have, if this is true for all graphs that have multiple of 5 and 6 numbers of vertices, in theory, there is a chance that they will also use the same amount of colors as they do vertices. With the math that was used for this research, a lower bound could not be found. We will consider this the beginning of a bigger research project.

References

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[2] Prosenjit Bose and Ferran Hurtado and Eduardo Rivera-Campo and David R. Wood, *Partitions of complete geometric graphs into plane trees*, Computational Geometry 34, 116–125 (2005)

[3] Alexandr Kostochka and Jan Kratochvíl, *Covering and coloring polygon-circle graphs*, Discrete Mathematics 163, 299–305 (1997).

[4] Nikita Chernega and Alexandr Polyanskii and Rinat Sadykov, *Disjoint edges in geometric graphs*